Nonconservative Evaluation of Uniform Stability Margins of Multivariable Feedback Systems

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This paper discusses concepts of stability margins of multivariable feedback systems. Independent and uniform stability margins are defined. A previous conjecture that the uniform margins may be computed by using the eigenvalue magnitudes instead of the singular values in the robust stability criteria is theorized. The nonconservatism provided by this theory in the evaluation of uniform margins is discussed, along with limitations of the uniform margins. Also presented is a method of using the uniform margins to extend the region of stability beyond what can be specified by singular values. Results are demonstrated numerically in an example of a lateral attitude control system for a drone aircraft.

Introduction

GAIN and phase margins have long been accepted as useful concepts in the specification of single-input single-output (SISO) feedback systems because they give the user of a control system a feel of how safe the system is so far as the stability is concerned. In extending these useful concepts to multiple-input multiple-output (MIMO) feedback systems, diversity and ambiguity often arise. The one-loop-at-a-time stability margins fail to account for the simultaneous variations in a MIMO feedback system and hence may be unacceptable as relative stability measures.[1] The norm-bounded robustness criteria[1,2] guarantee closed-loop stability but give the user no idea as to how the individual elements of the gain matrix may vary without destabilizing the closed-loop system. It is possible to obtain bounds on the magnitude of each element in the perturbation matrix of the loop transfer function for the stable operation of the feedback system.[3,4] In a general sense, these bounds are gain margins of the MIMO system. However, they are derived under the assumption that the phases and magnitudes of all elements in the perturbation matrix may vary simultaneously in the worst possible direction with unlimited phase variations. This is equivalent to having all the direct and crossfeed transfer functions varied independently and simultaneously. Since the worst possible variations are a mathematical extreme, these general gain margins are unduly conservative and, due to their uncorrelated multivariate nature, do not give a clear notion of how far the feedback system is from becoming unstable.

More meaningful stability margins may be defined[1] as limits within which the gains of all feedback loops may vary independently at the same time without destabilizing the system, while the phase angles remain at their nominal values, and vice versa. This amounts to setting the limits for independent gain or phase variations in a diagonal perturbation matrix for a multiplicative perturbation model. The zero off-diagonal elements in the perturbation matrix coordinate the variations of the crossfeed transfer functions and render the perturbation more tractable. However, these stability margins, as computed via the singular-value based robust stability criterion,[1] also tend to be very conservative. In an attempt to relax the conservatism in the evaluation of stability margins of a two-input two-output lateral attitude control system of a drone aircraft, Mukhopadhyay and Newsom[5] experimented with using the magnitudes of the eigenvalues instead of the singular values in the robustness criterion. By examining the Nyquist plot of the eigenvalues of the return difference matrix of the control system in their study, they conjectured that the "eigenvalue-based" gain (or phase) margins are limits within which the gains (or phases) of all feedback loops vary uniformly without destabilizing the feedback system while the phase angles (or gains) remain at their nominal values. In fact, since the spectral radius (maximum of the moduli of the eigenvalues) of a matrix is the greatest lower bound of all norms of that matrix, the conjecture of Ref. 5 is the least conservative for the evaluation of the uniform stability margins by means of norm-bounded robust stability criteria.

The uniform variations of multiloop gains and phases are also interesting in that the uniformity constraints give the multiloop variations a single-variable nature. Hence, the regions of stability in the gain and phase spaces degenerate into line segments. At each stable nominal operating point in the gain and phase spaces one such line segment may be constructed. In this fashion, the uniform gain and phase margins facilitate a nonconservative but discrete representation of the regions of stability in multidimensional gain and phase spaces.

In this paper, the conjecture given in Ref. 5 is proved. Uniformity in the gain and phase variations in feedback loops may be viewed as a special structure in the perturbation of an open-loop transfer matrix. Therefore, weighted ℓ_l and ℓ_∞ norms[3] are used in the norm-bounded stability robustness criteria to derive the formulas from which the uniform stability margins are computed. The concept and the one-dimensional characteristics of the uniform stability margins and their use in discretizing the regions of stability in multidimensional gain and phase spaces are demonstrated.

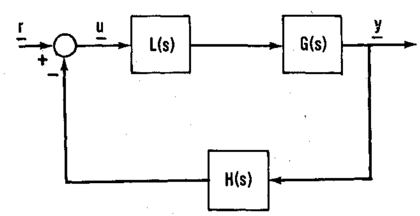

Fig. 1 Feedback system with input-multiplicative perturbations.

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The two-input two-output lateral attitude control system of a drone aircraft[5] is used again to demonstrate the extent of reduction of conservatism in determining the regions of stability and to demonstrate the feasibility of discretizing the regions of stability in multidimensional gain and phase spaces into line segments characterized by the uniform variation of loop gains and phases. The statement and proof of the conjecture are preceded by definitions of the commonly used[5,6] independent gain and phase margins and the proposed uniform gain and phase margins.

Stability Margins of Multivariable Feedback Systems

Definition 1: Independent gain margins are limits within which the gains of all feedback loops may vary independently at the same time without destabilizing the system, while the phase angles remain at their nominal values. Independent phase margins are limits within which the phase angles of all feedback loops may vary independently at the same time without destabilizing the system while the gains remain at their nominal values.

The independent gain and phase margins vary with the point at which the complex loop gains are measured. For a general nonunity feedback system as shown in Fig. 1, if the loop is broken at **u** to measure the complex loop gains, then the simultaneous perturbation in each loop may be represented by a diagonal perturbation matrix $L(s)$ preceding the plant $G(s)$. If the loop is to be broken at the output y, then $L(s)$ should be inserted before the feedback block $H(s)$. For $s=j\omega$, let $L(s)$ be

$$L(j\omega)=\mathrm{diag}[\beta_1(\omega)e^{j\theta_1(\omega)} \quad \beta_2(\omega)e^{j\theta_2(\omega)}\ldots\beta_n(\omega)e^{j\theta_n(\omega)}] \quad (1)$$

Independent gain margins are limits within which $\beta_i(\omega)$ may vary independently for each i withhout destabilizing the system, while $\theta_i(\omega)=0$ for all ω and all i. Independent phase margins are limits within which $\theta_i(\omega)$ may vary independently for each i without destabilizing the system, while $\beta_i(\omega)=1$ for all ω and all i.

One also can let both $\beta_i(\omega)$ and $\theta_i(\omega)$ vary simultaneously and independently for each i. The limits within which $\beta_i(\omega)$ may vary depend upon $\theta_i(\omega)$ and vice versa, and hence are unwieldy for use as gain and phase margins.

Definition 2: Uniform gain margins are limits within which the gains of all feedback loops may vary uniformly at the same time without destabilizing the system, while the phase angles remain at their nominal values. Uniform phase margins are limits within which the phase angles of all feedback loops may vary uniformly at the same time without destabilizing the system, while the gains remain at their nominal values.

For the system of Fig. 1, let $L(j\omega)$ be given by

$$L(j\omega)=\beta(\omega)e^{j\theta(\omega)}K(j\omega) \quad (2)$$

where $K(j\omega)$ is the nominal complex loop gain matrix given by

$$K(j\omega)=\mathrm{diag}[k_1(j\omega) \quad k_2(j\omega)\ldots k_n(j\omega)] \quad (3)$$

Uniform gain margins with respect to the nominal loop gain $K(j\omega)$ are limits within which $\beta(\omega)$ may vary without destabilizing the feedback system while $\theta(\omega)=0$ for all ω. Uniform phase margins with respect to the nominal loop gain $K(j\omega)$ are limits within which $\theta(j\omega)$ may vary without destabilizing the feedback system while $\beta(\omega)=1$ for all ω.

The nominal gain $K(j\omega)$ also represents a nominal operating point in the gain and phase spaces about which the system is uniformly perturbed. Since there is only one complex variable in Eq. (2), the regions of stability about each stable nominal operating point $K(j\omega)$ in the gain and phase spaces are straight-line segments. These one-dimensional regions facilitate a discrete representation of the regions of stability in multidimensional spaces. This will be demonstrated in an example after the formulas for nonconservative evaluation of the uniform stability margins are derived.

Nonconservative Evaluation of Uniform Stability Margins

For the system of Fig. 1, two robust stability criteria can be written; i.e., for all s on the Nyquist contour

$$\bar{\sigma}[L(s)-I]<\underline{\sigma}(I+\{H(s)G(s)\}^{-1}) \quad (4)$$

and

$$\bar{\sigma}[L^{-1}(s)-I]<\alpha\leq\underline{\sigma}(I+H(s)G(s)] \quad (5)$$

where $\alpha\leq 1$, $\underline{\sigma}(\cdot)$ is the maximum singular value of the matrix in the argument, $\sigma(\cdot)$ the minimum singular value, and I the identity matrix. Since these criteria are derived[1,2] on the basis of multivariable Nyquist theory, some preliminary conditions on the nominal and perturbed systems must hold. These conditions are[1,6]: a) the open-loop characteristic polynomials of the nominal system and the perturbed system must have the same number of closed right-half plane roots, b) all imaginary poles of the open-loop perturbed system must also be poles of the open-loop nominal system, and c) the nominal system must be closed-loop stable. Since the right-hand sides of inequalities (4) and (5) are measures of the nearness of $\{H(s)G(s)\}^{-1}$ and $H(s)G(s)$ to the critical point of stability, inequalities (4) and (5) may be referred to as the inverse Nyquist formulation and the Nyquist formulation, respectively.

If these formulations are employed to determine the stability margins, the results obtained are always conservative. However, these criteria are special cases of inequalities involving general matrix norms,[3,6,7] namely,

$$\|L(s)-I\|<1/\|[I+\{H(s)G(s)\}^{-1}]^{-1}\| \quad (6)$$

and

$$\|L^{-1}(s)-I\|<\alpha\leq 1/\|[I+H(s)G(s)]^{-1}\| \quad (7)$$

where $\alpha\leq 1$ and the vertical double bars $\|\cdot\|$ denote general matrix norms which include the maximum singular values as a special case. Inequality (6) follows from a simple generalization of the derivation given in Ref. 2, and has been used in other papers.[4,7] Inequality (7) can be derived in the same fashion as the derivation of Inequality (5) (see Ref. 1). Nevertheless, because the error matrix $L^{-1}(s)-I$ appears nonlinearly in the convex combination of nominal and perturbed loop transfer matrices, the generalization of the proof in Ref. 1 to the proof of Inequality (7) is not trivial. The proof of Inequality (7) can be found in a recent paper.[8] It is

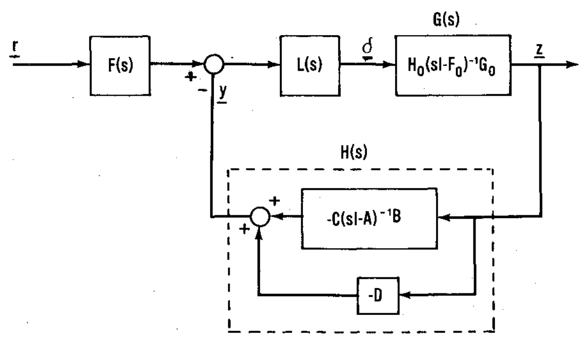

Fig. 2 Lateral attitude control system of a drone aircraft.

evident that the conservatism of robust stability criteria may be reduced by using Inequality (6) or (7) instead of Eqs. (4) and (5), respectively. This reduction will occur if for all s on the Nyquist contour the norms of $L(s)-I$ and $[I+\{H(s)G(s)\}^{-1}]^{-1}$, or $L^{-1}(s)-I$ and $[I+H(s)G(s)]^{-1}$ can be computed to be less than their respective maximum singular values. Note that $\underline{\sigma}(A)=1/\bar{\sigma}(A^{-1})$.

The conjecture of Mukhopadhyay and Newsom[5] states that if $L(s)$ is characterized by Eq. (2) and if $K(j\omega)=I$, the maximum and minimum singular values in Inequality (5) may be replaced by the maximum and minimum magnitudes of the eigenvalues, respectively. More specifically, the conjecture states that under the above assumption, the stability of the system of Fig. 1, and hence the stability margins for $K(j\omega)=I$, may be determined by the inequality

$$\max_i |\lambda_i[L^{-1}(s)-I]| < \alpha \leq \min_i |\lambda_i[I+H(s)G(s)]| \tag{8}$$

for $\alpha \leq 1$, and for all s on the Nyquist contour, where $\lambda_i(\cdot)$ denotes the ith eigenvalue of the matrix in the argument. Since any norm of a matrix is always greater than or equal to the spectral radius of the matrix, Inequality (8) is the least conservative of all computations of Inequality (7), which includes Eq. (5) as a special case.

The subsequent development theorizes the above conjecture in a general framework, provides a proof, and then derives the nonconservative formulas of uniform stability margins. The same conjecture can also be applied to Eq. (4), and will also be proved here. In the formulation of uniform variations of complex multiloop gains, the nominal gain matrix $K(j\omega)$ of Eq. (2) is not an identity matrix in general and is given a priori. It may be treated as part of the plant. Therefore, the resulting norm-bounded robust stability criteria for the system of Fig. 1 with $L(s)$ characterized by Eq. (2) may be written as

$$\|[\beta(\omega)e^{j\theta(\omega)}-1]I\| < 1/\|[I+\{H(j\omega)G(j\omega)K(j\omega)\}^{-1}]^{-1}\| \tag{9}$$

$$\left\|\left[\frac{1}{\beta(\omega)}e^{-j\theta(\omega)}-1\right]I\right\| < \alpha \leq \frac{1}{\|[I+H(j\omega)G(j\omega)K(j\omega)]^{-1}} \tag{10}$$

for $0\leq\omega<\infty$. These inequalities are obtained by substituting $j\omega$ for s, $G(j\omega)K(j\omega)$ for $G(j\omega)$, and $\beta(\omega)\exp[j\theta(\omega)]$ for $L(j\omega)$ in Inequalities (6) and (7). Practical systems with $H(s)G(s)K(s)\to 0$ as $s\to\infty$ are assumed.

Lemma 1. Let A be an $n\times n$ matrix, $|A|$ be the nonnegative matrix formed by taking the absolute values of the elements of A, and $Q=\mathrm{diag}[q_1 q_2 \ldots q_n]$ be an $n\times n$ diagonal matrix with $q_i>0$, for $i=1,2,\ldots,n$. The maximum row sum of the matrix product $Q^{-1}|A|Q$ is a norm of A, called the Q-weighted ℓ_∞ norm, denoted by

$$\|A\|_{Q\infty} = \text{max row sum of } Q^{-1}|A|Q \tag{11}$$

The proof of Lemma 1 may be found in Ref. 9.

Lemma 2. Given any nonsingular $n\times n$ matrix M, and any $n\times n$ matrix A, if $f(A)$ is a norm of A, then $f(M^{-1}AM)$ is also a norm of A.

This Lemma is also given in Ref. 9.

Theorem 1. If the preliminary conditions following Eqs. (4) and (5) hold, and $L(s)$ is characterized by (2) [$K(j\omega)$ need not be diagonal], then the system of Fig. 1 is stable if either one of the following inequalities is satisfied:

$$|\beta(\omega)e^{j\theta(\omega)}-1| < \min_i|\lambda_i[I+\{H(j\omega)G(j\omega)K(j\omega)\}^{-1}]| \tag{12}$$

$$|[1/\beta(\omega)]e^{-j\theta(\omega)}-1| < \alpha \leq \min_i|\lambda_i[I+H(j\omega)G(j\omega)K(j\omega)]| \tag{13}$$

for $\alpha\leq 1$, and for all $0\leq\omega<\infty$.

Proof: Let $M(s)$ be the modal matrix of $[I+\{H(j\omega)G(j\omega)K(j\omega)\}^{-1}]^{-1}$. Then $M^{-1}(j\omega)[I+\times\{H(j\omega)G(j\omega)K(j\omega)\}^{-1}]^{-1}M(j\omega)$ is in Jordan canonical form, and all of its elements below the diagonal are zero. For notational brevity, let this Jordan canonical form be denoted by $J(j\omega)$, i.e.,

$$J(j\omega) = M^{-1}(j\omega)[I+\{H(j\omega)G(j\omega)K(j\omega)\}^{-1}]^{-1}M(j\omega) \tag{14}$$

Let

$$Q=\mathrm{diag}[1\epsilon\ \epsilon^2\ldots\epsilon^{n-1}] \tag{15}$$

where ϵ is an arbitrarily small positive number. Then, by virtue of Lemmas 1 and 2, a special form of Eq. (9) may be written as

$$\|M^{-1}(j\omega)[\{\beta(\omega)e^{j\theta(\omega)}-1\}I]M(j\omega)|_{Q\infty} < 1/\|J(j\omega)\|_{Q\infty} \tag{16}$$

The left side of the above inequality is readily computed to be $|\beta(\omega)e^{j\theta(\omega)}-1|$. With the aid of Eqs. (11), (14), and (15), the Q-weighted ℓ_∞ norm of the Jordan canonical form of $J(j\omega)$ is found to be

$$\|J(j\omega)\|_{Q\infty} = \max_i|\lambda_i[(I+\{H(j\omega)G(j\omega)K(j\omega)\}^{-1})^{-1}]| + \delta\epsilon \tag{17}$$

The variable δ in Eq. (17) is either 1 or 0, depending on the superdiagonal elements in $J(j\omega)$. Thus, Eq. (16) is equivalent to

$$|\beta(\omega)e^{j\theta(\omega)}-1| < 1/\max_i|\lambda_i[(I+\{H(j\omega)G(j\omega)K(j\omega)\}^{-1})^{-1}]| + \delta\epsilon \tag{18}$$

However, for any invertible A, any eigenvalue of the inverse of A is the inverse of an eigenvalue of A. Therefore, Inequality (18) is equivalent to

$$|\beta(\omega)e^{j\theta(\omega)}-1| < \frac{\min_i|\lambda_i[I+\{H(j\omega)G(j\omega)K(j\omega)\}^{-1}]|}{1+\delta\epsilon\min_i|\lambda_i[I+\{H(j\omega)G(j\omega)K(j\omega)\}^{-1}]|} \tag{19}$$

which may be written as

$$|\beta(\omega)e^{j\theta(\omega)}-1| < \min_i|\lambda_i[I+\{H(j\omega)G(j\omega)K(j\omega)\}^{-1}]| - 0(\epsilon) \tag{20}$$

where $0(\epsilon)$ vanishes with ϵ in such a way that $0\leq 0(\epsilon)<k\epsilon$ for some $k>0$ and for sufficiently small ϵ. This proves Inequality (12). Inequality (13) follows from Inequality (10) by the same reasoning. Q.E.D.

Corollary 1.1. If there exists $a_o>0$ such that for $0\leq\omega<\infty$,

$$\min_i|\lambda_i[I+\{H(j\omega)G(j\omega)K(j\omega)\}^{-1}]| \geq a_o \tag{21}$$

then the uniform gain margins of the MIMO system of Fig. 1 are given by

$$1-a_o < \beta(\omega) < 1+a_o \tag{22}$$

and the uniform phase margins are

$$-\pi \leq \theta(\omega) \leq \pi \quad \text{if} \quad 2 < a_o \tag{23}$$

$$-2\sin^{-1}(a_o/2) < \theta(\omega) < 2\sin^{-1}(a_o/2) \quad \text{if} \quad a_o \leq 2 \tag{24}$$

Proof: In view of Inequalities (21) and (12), the system of Fig. 1 is stable if

$$|\beta(\omega)e^{j\theta(\omega)} - 1| < a_o \tag{25}$$

Letting $\theta(\omega) = 0$ in Inequality (25) gives Inequality (22). To obtain the uniform phase margins, let $\beta(\omega) = 1$ in Inequality (25) to yield

$$|e^{j\theta(\omega)} - 1| < a_o \tag{26}$$

Conditions (23) and (24) are a result of Inequality (26). Q.E.D.

Corollary 1.2. If there exists some $\alpha_0 \leq 1$ such that for $0 \leq \omega < \infty$,

$$\min_i |\lambda_i[I + H(j\omega)G(j\omega)K(j\omega)]| \geq \alpha_0 \tag{27}$$

then the uniform gain and phase margins of the MIMO system of Fig. 1 are given by

$$1/(1+\alpha_0) < \beta(\omega) < 1/(1-\alpha_0) \tag{28}$$

and

$$-2\sin^{-1}(\alpha_0/2) < \theta(\omega) < 2\sin^{-1}(\alpha_0/2) \tag{29}$$

respectively.

Proof: Letting $\theta(\omega) = 0$ and $\alpha = \alpha_0$ in (13) yields Formula (28) for the uniform gain margins. Letting $\beta(\omega) = 1$ and $\alpha = \alpha_0$ in Inequality (13) yields Formula (29) for the uniform phase margins. Q.E.D.

Theorem 1 and the Corollaries may be restated for independent gain and phase margins by substituting singular values for eigenvalues, $\beta_i(\omega)$ for $\beta(\omega)$, and $\theta_i(\omega)$ for $\theta(\omega)$, because the maximum singular value of a diagonal matrix is the magnitude of its largest element. However, in many cases less conservative results can be achieved by using norm measures other than singular values. For the sake of covenience, Corollaries 1.1 and 1.2 may be referred to as the inverse Nyquist formulation and the Nyquist formulation of uniform stability margins, respectively. Note that the inversion of the loop transfer function matrix in the inverse Nyquist formulation may be avoided by substituting $[I+H(s)G(s)]^{-1}H(s)G(s)$ for $[I+\{H(s)G(s)\}^{-1}]^{-1}$ in the right side of Inequality (6) and then rewriting Inequalities (12), (21), and (27) accordingly.

Since multivariable stability margins are based on sufficient conditions, the union of the stability regions given by different methods is also a valid region of stability. When computing uniform stability margins, the nominal system [when $L(s) = K(s)$] is required to be stable in order for the robust stability criteria to be valid. The selection of the nominal gain $K(s)$ may be aided by the formulas for independent gain margins. Thus, the combined use of independent and uniform stability margins enables one to extend beyond the conservative regions of stability established by the independent gain and phase margins along selected straight lines in the gain and phase spaces. This is demonstrated in the example in the next section.

Regions of Stability in the Gain and Phase Spaces

As in the SISO case, stability margins of a MIMO system guarantee the stability when either the gains or phases, but not both, of all the feedback loops may vary within the prescribed limits without destabilizing the closed-loop system. Therefore, if the uniform gain margin of a MIMO system at a nominal gain $K(j\omega)$ is $[g_{m1}, g_{m2}]$, then the system (Fig. 1) is stable when

$$L(j\omega) = \beta_0 K(j\omega) \tag{30}$$

for all β_0 satisfying $g_{m1} < \beta_0 < g_{m2}$. To determine the region of stability in the gain space where phase angles of all feedback loops are assumed unperturbed, all elements in $K(j\omega)$ are selected to be real constants, i.e.,

$$K(j\omega) = \text{diag}[k_1 \; k_2 \ldots k_n] \tag{31}$$

The coordinates of the gain space are loop gains [magnitudes of the elements of $L(j\omega)$ given by Eqs. (30) and (31)] β_i, where $\beta_i = \beta_0 k_i$, for $i = 1,2,\ldots,n$. Thus, the region of stability specified by the uniform gain margin is a line segment between points $g_{m1}\boldsymbol{k}$ and $g_{m2}\boldsymbol{k}$ in the gain space, where $\boldsymbol{k}$ is the vector

$$\boldsymbol{k} = (k_1, k_2, \ldots, k_n) \tag{32}$$

In the phase space, the absolute gains of all feedback loops are assumed unperturbed. To determine the region of stability in the phase space, all elements in $K(j\omega)$ must be selected to be complex constants of unity magnitude, i.e.,

$$K(j\omega) = \text{diag}[e^{j\phi_1} e^{j\phi_2} \ldots e^{j\phi_n}] \tag{33}$$

Thus, if the uniform phase margin of a MIMO system at a nominal gain $K(j\omega)$ given by Eq. (33) is $[\phi_{m1}, \phi_{m2}]$ then the system is stable when

$$L(j\omega) = e^{j\theta_0} K(j\omega) \tag{34}$$

for all θ_0 satisfying $\phi_{m1} < \theta_0 < \phi_{m2}$. The coordinates of the phase space are loop phases [phase angles of $L(j\omega)$ given by Eqs. (33) and (34)] θ_i, where $\theta_i = \phi_i + \theta_0$, for $i = 1,2,\ldots,n$. Thus, the region of stability specified by the uniform phase margin is a line segment between $\boldsymbol{\phi} + \phi_{m1}\boldsymbol{e}$ and $\boldsymbol{\phi} + \phi_{m2}\boldsymbol{e}$ in the phase space, where $\boldsymbol{e}$ and $\boldsymbol{\phi}$ are the n-vectors given by

$$\boldsymbol{e} = (1 \; 1 \ldots 1) \tag{35}$$

$$\boldsymbol{\phi} = (\phi_1 \phi_2 \ldots \phi_n) \tag{36}$$

In contrast with the uniform stability margins, the regions of stability prescribed by independent stability margins are hypercubes in the gain and phase spaces. It is easy to see from Definition 1 that if the independent gain margin of a MIMO system is $[g_a, g_b]$, the system is stable when

$$L(j\omega) = \text{diag}[\beta_1 \beta_2 \ldots \beta_n] \tag{37}$$

with $g_a < \beta_i < g_b$ for $i = 1,2,\ldots,n$. If the independent phase margin of a MIMO system is $[\phi_a, \phi_b]$, the system is stable when

$$L(j\omega) = \text{diag}[e^{j\theta_1} \; e^{j\theta_2} \ldots e^{j\theta_n}] \tag{38}$$

with $\phi_a < \theta_i < \phi_b$ for $i = 1,2,\ldots,n$.

Since the operating points inside and on the boundary of the hypercube of the independent stability margins are guaranteed to be stable,[8] they may be used as nominal gains $K(j\omega)$ in Eqs. (30) and (34) to determine the one-dimensional regions of stability in the n-dimensional gain and phase spaces.

Example: For the purpose of comparison, the eighth order lateral attitude control system of a drone aircraft used in Ref. 5 is used here. The block diagram of the system is given in Fig.

2 and the numerical data of Fig. 2 are given in Table 1. The perturbation matrix $L(s)$ is characterized by

$$L(j\omega) = \begin{bmatrix} \beta_1 e^{j\theta_1} & 0 \\ 0 & \beta_2 e^{j\theta_2} \end{bmatrix} \tag{39}$$

where β_1, β_2, θ_1, and θ_2 are constants in the gain and phase margin calculations.

The graphs of the minimum magnitudes of the eigenvalues and singular values of $I+H(j\omega)G(j\omega)$ and $I+\{H(j\omega)G(j\omega)\}^{-1}$ are plotted vs frequency in Figs. 3 and 4. The minimum values of these curves are found to be

$$\alpha_0 = \min_{\omega} \min_{i} |\lambda_i[I+H(j\omega)G(j\omega)]| = 0.6494 \tag{40}$$

$$\alpha_0' = \min_{\omega} \underline{\sigma}[I+H(j\omega)G(j\omega)] = 0.2463 \tag{41}$$

$$a_o = \min_{\omega} \min_{i} |\lambda_i[I+\{H(j\omega)G(j\omega)\}^{-1}]| - 0.4417 \tag{42}$$

$$a_o' = \min_{\omega} \underline{\sigma}\{I+\{H(j\omega)G(j\omega)\}^{-1}\} = 0.2279 \tag{43}$$

The independent gain and phase margins (IGM and IPM, respectively), may be calculated from α_0' and a_o' as

$$\mathrm{IGM} = [1/(1+\alpha_0'), 1/(1-\alpha_0')] = [0.8024, 1.3268] \tag{44}$$

$$\mathrm{IGM} = [1-a_o', 1+a_o'] = [0.7721, 1.2279] \tag{45}$$

$$\mathrm{IPM} = [-2\sin^{-1}(\alpha_0'/2), 2\sin^{-1}(\alpha_0'/2)] = [-14.147\,\mathrm{deg}, 14.147\,\mathrm{deg}] \tag{46}$$

$$\mathrm{IPM} = [2\sin^{-1}(a_o'/2), 2\sin^{-1}(a_o'/2)] = [13.09\,\mathrm{deg}, 13.09\,\mathrm{deg}] \tag{47}$$

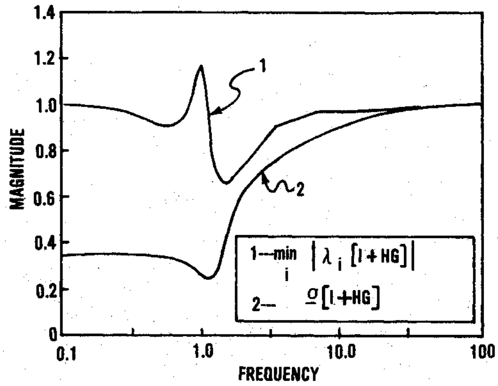

Fig. 3 Minimum eigenvalue and $\underline{\sigma}$ of return difference matrix of Fig. 2 when $L(j\omega)=I$.

Table 1 Numerical data for the system of Fig. 2

$$A=\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C=\begin{bmatrix} 0.1491 & 0 \\ 0 & -4.116 \end{bmatrix} \quad D=\begin{bmatrix} 0 & 0 \\ 0 & 2.058 \end{bmatrix}$$

$$F(s)=\begin{bmatrix} 0.1491/s & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_0=\begin{bmatrix} -0.0827 & -0.1423\times 10^{-3} & -0.9994 & 0.0414 & 0 & 0.1862 \\ -46.86 & -2.757 & 0.3896 & 0 & -124.3 & 128.6 \\ -0.4248 & -0.06224 & -0.0671 & 0 & -8.792 & -20.46 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20. & 0 \\ 0 & 0 & 0 & 0 & 0 & -20.0 \end{bmatrix}$$

$$\lambda(F_0)=\begin{bmatrix} -0.03701 & \textit{spiral mode} \\ 0.1889\pm j1.051 & \text{dutch roll} \\ -3.25 & \text{roll convergence} \\ -20.0 & \text{elevon actuator} \\ -20.0 & \text{rudder actuator} \end{bmatrix}$$

$$G_0=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad H_0=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.07 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Note that the union of the regions of stability found by any sufficient stability criteria is contained in the actual region of stability. Hence, the gain margins of Eqs. (44) and (45) may be combined. Similarly, the phase margins of Eqs. (46) and (47) may also be combined, but the right side of Eq. (47) is already contained in that of Eq. (46). Thus

$$\mathrm{IGM} = [0.7721, 1.3268] \tag{48}$$

$$\mathrm{IPM} = [-14.147\,\mathrm{deg}, 14.147\,\mathrm{deg}] \tag{49}$$

The regions of stability represented by Eqs. (48) and (49) are shown as squares in the gain and phase planes in Figs. 5 and 6, respectively. These are the regions of stability specified by singular value robust stability criteria for independent loop gain variations when phase angles are kept at nominal values, and for independent loop phase variations when loop gains are kept at nominal values. Each point (β_1, β_2) in the gain plane of Fig. 5 represents an operating point of the system of Fig. 2, when $L = \mathrm{diag}[\beta_1, \beta_2]$. Similarly, each point (θ_1, θ_2) in the phase plane of Fig. 6 represents an operating point of the system of Fig. 2, when $L = \mathrm{diag}[e^{j\theta_1}, e^{j\theta_2}]$. If the system operates anywhere inside the square ABCD in Fig. 5 or PQRS in Fig. 6, it is stable.

The uniform gain and phase margins [where $\beta_1 = \beta_2 = \beta$, and $\theta_1 = \theta_2 = \theta$ in Eq. (39)] for which $K(j\omega) = I$ are calculated from α_0 and a_o of Eqs. (40) and (42). These are found to be (0.5583, 2.8523) and (−37.895 deg, 37.895 deg), respectively. It is seen that these margins are much larger (less conservative) than those in Eqs. (48) and (49), but yield only line segments in the gain and phase spaces vs the squares ABCD and PQRS in Figs. 5 and 6 obtained through independent margins. However, these line segments obtained by uniform margins can be used to extend the regions of stability in certain directions considerably beyond what can be established by independent margins.

To demonstrate this use of uniform gain and phase margins, let the nominal system be operating at point A (Fig. 5) and assume uniform perturbations. The uniform gain margins may be found by formulas (22) and (28) using

$$K(j\omega) = K_a = \begin{bmatrix} 0.7721 & 0 \\ 0 & 1.3268 \end{bmatrix} \tag{50}$$

The graphs of the minimum magnitude of the eigenvalues of $I + H(j\omega)G(j\omega)K_a$ and $I + \{H(j\omega)G(j\omega)K_a\}^{-1}$ are plotted vs frequency in Fig. 7. The minimum value of these curves are found to be

$$\alpha_0 = \min_{\omega} \min_{i} |\lambda_i[I + H(j\omega)G(j\omega)K_a]| = 0.9094 \tag{51}$$

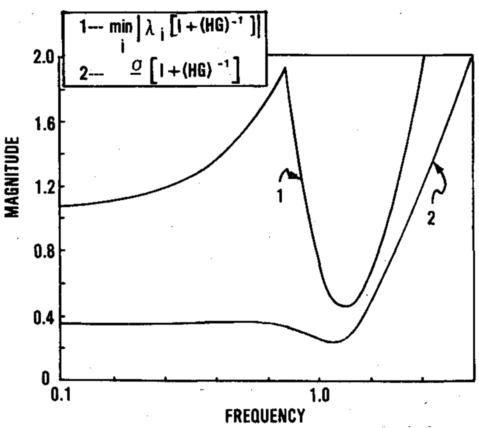

Fig. 4 Minimum eigenvalue and $\underline{\sigma}$ of inverse return difference matrix of Fig. 2 when $L(j\omega) = I$.

$$a_o = \min_{\omega} \min_{i} |\lambda_i[I + \{H(j\omega)G(j\omega)K_a\}^{-1}]| = 0.6058 \tag{52}$$

The uniform gain margins based on the Nyquist and inverse Nyquist formulations are found to be

$$\mathrm{UGM} = [1/(1+\alpha_0), 1/(1-\alpha_0)] = [0.5237, 11.0374] \tag{53}$$

$$\mathrm{UGM} = [1 - a_o, 1 + a_o] = [0.3942, 1.6058] \tag{54}$$

respectively. The combined UGM is

$$\mathrm{UGM} = [0.3942, 11.0374] \tag{55}$$

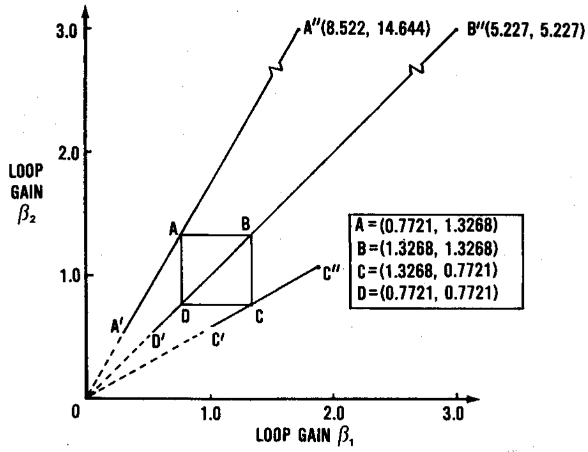

Fig. 5 Gain-plane region of stability for the system of Fig. 2.

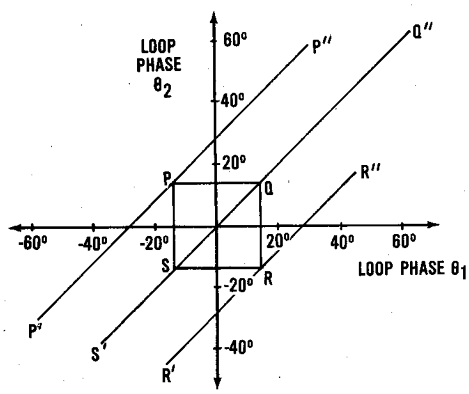

Fig. 6 Phase-plane region of stability for the system of Fig. 2.

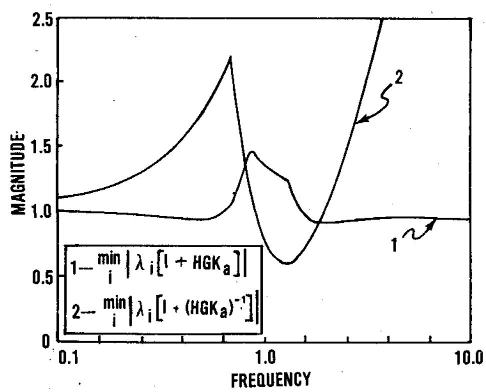

Fig. 7 Minimum eigenvalues when the system is operating at A in Fig. 5.

The combined UGM of Eq. (55) specifies that the operating points on the line segment $A'AA''$ in the gain plane (Fig. 5) are stable, where

$$\overline{OA'}/\overline{OA}=0.3942 \tag{56}$$

$$\overline{OA''}/\overline{OA}=11.0374 \tag{57}$$

where the upper bar denotes the length of the line segment. Applying similar computations to points *B*, *C*, and *D* in Fig. 5 shows that the operating points on the line segments $D'DBB''$ and $C'CC''$ are stable operating points, where

$$\overline{OD'}/\overline{OD}=0.7018 \tag{58}$$

$$\overline{OB''}/\overline{OB}=3.9397 \tag{59}$$

$$\overline{OC'}/\overline{OC}=0.7718 \tag{60}$$

$$\overline{OC''}/\overline{OC}=1.4146 \tag{61}$$

Table 2 shows the values of α_0 and a_o at points *B*, *C*, and *D*. Note that the points B' and D'' are not shown in Fig. 5 as they lie between the endpoints B'' and D'.

On the phase plane, let the system be operating at point *P* and assume uniform perturbations. The uniform phase margins may be found by formulas (23), (24), and (29), using

$$K(j\omega)=K_p=\begin{bmatrix} e^{-j14.147^\circ} & 0 \\ 0 & e^{j14.147^\circ} \end{bmatrix} \tag{62}$$

The graphs of the minimum magnitude of the eigenvalues of $I+H(j\omega)G(j\omega)K_p$ and $I+\{H(j\omega)G(j\omega)K_p\}^{-1}$ are plotted vs frequency in Fig. 8. The minimum values of these curves are found to be

$$\alpha_0''=\min_\omega \min_i |\lambda_i[I+H(j\omega)G(j\omega)K_p]|=0.7506 \tag{63}$$

$$a_0''=\min_\omega \min_i |\lambda_i[I+\{H(j\omega)G(j\omega)K_p^{-1}\}]|=0.4550 \tag{64}$$

The uniform phase margins (UPM) based on the Nyquist and inverse Nyquist formulations, respectively, are found to be

$$\text{UPM}=[-2\sin^{-1}(\alpha_0''/2),2\sin^{-1}(\alpha_0''/2)]$$
$$=[-44.085\text{ deg}, 44.085\text{ deg}] \tag{65}$$

$$\text{UPM}=[-2\sin^{-1}(a_o''/2),2\sin^{-1}(a_o''/2)]$$
$$=[-26.301\text{ deg}, 26.301\text{ deg}] \tag{66}$$

The region of stability specified by Eq. (66) is contained inside the one specified by Eq. (65). Hence, the uniform phase margin is given by Eq. (65). Thus, the operating points on the line segment $P'PP''$ in the phase plane (Fig. 6) are stable, with

$$\overline{P'P}=\overline{PP''}=\sqrt{2}\times 44.085\text{ deg} \tag{67}$$

Similar computations on points *Q*, *R*, and *S* show that operating points on line segments $S'SQQ''$ and $R'R'R''$ are stable operating points, with

$$\overline{S'S}=\sqrt{2}\times 23.570\text{ deg} \tag{68}$$

$$\overline{QQ''}=\sqrt{2}\times 48.634\text{ deg} \tag{69}$$

$$\overline{R'R}=\overline{RR''}=\sqrt{2}\times 30.510\text{ deg} \tag{70}$$

Table 2 Minimum eigenvalues for points in Figs. 5 and 6

Point	α_0	a_o
B	0.7462	0.5648
C	0.2931	0.2282
D	0.4003	0.2982
Q	0.8236	0.4850
R	0.5262	0.3974
S	0.4085	0.3281

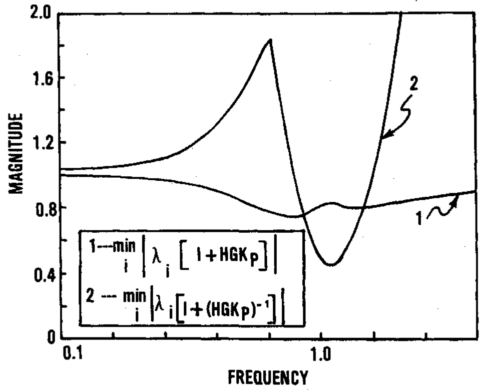

Fig. 8 Minimum eigenvalues when the system is operating at *P* in Fig. 6.

Table 2 shows the values of α_0 and a_o at points *Q*, *R*, and *S*. Once again, Q' and S'' are not shown in Fig. 6 as they lie between Q'' and S'. The regions of stability in the gain and phase planes are thus extended considerably beyond the squares specified by the singular values along selected straight-line segments with the aid of uniform stability margins.

Note that near the actual boundary of stability, where the minimum singular value is small, ill-conditioning may be present and eigenvalue computations may be inaccurate. However, in that area, both the minimum singular value and the minimum eigenvalue are near zero, and there the singular value is preferable.

Conclusions

The concept of uniform stability margins is developed on the basis of uniform variations of multiloop gains and phases. It is proved that uniform stability margins may be computed by substituting modulii of eigenvalues for singular values in the singular-value-bounded robust stability criteria. This is the least conservative computation that is possible when a norm-bound robust stability criterion is used.

Regions of stability in the gain and phase spaces as specified by uniform stability margins are line segments which pass through the given nominal operating points. The uniform stability margins may be used to extend the regions of stability beyond what can be specified by the singular values of the return difference matrix or the inverse return difference matrix along selected straight lines in the least conservative manner.

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